A result on the phase diagram of a Ginzburg-Landau problem

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Abstract

Working with a particular modelization of Ginzburg-Landau phenomenological theory (see [Du01], [Du99] and Section II), we first recall the form of the phase diagram of this modelization as it usually drawn in the physical literature ([T], [Ki], [SST] and [Ge]).

We then study in detail the special case, when the critical Ginzburg Landau parameter k is equal to $\frac{1}{\sqrt{2}}$. This allows us to prove that the critical magnetic field $H_{c1}(k)$ is strictly decreasing at $k = \frac{1}{\sqrt{2}}$.

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I INTRODUCTION

In 1950 V. Ginzburg and L. Landau ([GL50]) have proposed a modelization for describing the various states of a superconducting material. They introduce a functional depending on a *wave function* ϕ and a *magnetic potential vector* $\mathbf{A}$, whose local minima will describe the properties of the material; in this modelization $|\phi|^2$ represents the local density of superconducting electrons.

Abrikosov ([Ab]) has introduced a particular Ginzburg-Landau modelization, which predicts the periodic structure for the zeros of ϕ, which was subsequently observed in experiments. His model depends on two positive parameters k and H_{ext}, called *Ginzburg-Landau parameter* and *external magnetic field*. It also assumed that:

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1. The superconductor is infinite, homogeneous and isotrop.
2. The magnetic field $\mathbf{H}_{\mathsf{ext}} = (0, 0, H_{\mathsf{ext}})$ is constant.
3. The energy functional $F(\phi, \mathbf{A})$ has a Ginzburg-Landau form and depends on the *Ginzburg-Landau parameter k*.
4. The pairs $(\phi, \mathbf{A})$ considered are gauge invariant along the z-axis and also along a lattice of $\mathbb{R}^2$.
5. The lattice has a fixed shape and there is one quantum flux per unit cell of it.

After some change of variable, recalled in Section II, we obtain the following formulation of the problem:

Denote $\mathcal{L}$ a lattice of $\mathbb{R}^2$, with fundamental domain Ω of area 1. Define the vector bundle E_1 over $\mathbb{R}^2/\mathcal{L}$ as the vector bundle, whose C^∞ sections are described by

$$C^\infty(E_1) = \left\{ \begin{array}{c} u : \mathbb{R}^2 \to \mathbb{C} \text{ s.t. } \forall (x,y) \in \mathbb{R}^2, \forall v = (v_x, v_y) \in \mathcal{L}, \\ u((x,y)+v) = e^{i\pi(v_x y - v_y x)} u(x,y) \end{array} \right\}$$

The vector bundle E_1 is non-trivial; this implies that any section $u \in C^\infty(E_1)$ has at least one zero in $\mathbb{R}^2/\mathcal{L}$.

The potential vector $\mathbf{a}$ belongs to the space

$$\{\mathbf{a} \in H^1_{\mathsf{loc}}(\mathbb{R}^2; \mathbb{R}^2) \text{ such that } \operatorname{div} \mathbf{a} = 0,\ \mathbf{a} \text{ is } \mathcal{L}\text{-periodic and } \int_\Omega \mathbf{a} = 0\}$$

We denote by $\mathcal{A}$ the space of all pairs $(u, \mathbf{a})$ with u being a H^1_{loc} section of E_1 and $\mathbf{a}$ belonging to the above space.

Denote H_{int} the *internal magnetic field* and $E_{k,H_{\mathsf{int}}}$ the functional defined over $\mathcal{A}$ by

$$E_{k,H_{\mathsf{int}}}(u, \mathbf{a}) \;=\; \int_\Omega \tfrac{\mu}{2} \|i\nabla u + (\mathbf{A}_0 + \mathbf{a})u\|^2 + \tfrac{1}{4}(1 - |u|^2)^2 + \tfrac{\mu^2 k^2}{2} |\operatorname{curl} \mathbf{a}|^2$$

with $\mu = \frac{H_{\mathsf{int}}}{2\pi k}$ and $\mathbf{A}_0 = \pi \begin{pmatrix} -y \\ x \end{pmatrix}$. We then define the energy of the superconductor as

$$E_{k,H_{\mathsf{ext}}}(H_{\mathsf{int}}, u, \mathbf{a}) = E_{k,H_{\mathsf{int}}}(u, \mathbf{a}) + \frac{1}{2}(H_{\mathsf{int}} - H_{\mathsf{ext}})^2.$$

The term $\frac{1}{2}(H_{\mathsf{int}} - H_{\mathsf{ext}})^2$ is a simple magnetic energy, while the term $E_{k,H_{\mathsf{int}}}$ is the internal energy of the superconductor. The energy $\mathcal{E}_{k,H_{\mathsf{ext}}}$ is then defined as the minimum of $E_{k,H_{\mathsf{ext}}}$ over all magnetic field H_{int} and pairs $(u, \mathbf{a}) \in \mathcal{A}$. Also we denote $m_E(k, H_{\mathsf{int}})$ the infimum of $E_{k,H_{\mathsf{int}}}$ over all pairs $(u, \mathbf{a}) \in \mathcal{A}$.

For $u = 0$, $\mathbf{a} = 0$ and $H_{\mathsf{int}} = H_{\mathsf{ext}}$ one obtains the energy $E_{\mathcal{N}} = \frac{1}{4}$, which is the energy of the so called *normal state*. In the limit case $H_{\mathsf{int}} = 0$, one obtains (see [Du99] or [Du01]) the energy $E_{\mathcal{P}} = \frac{H_{\mathsf{ext}}^2}{2}$, which is the energy of the *pure state*. This leads us to introduce three sets in $\mathbb{R}_+^* \times \mathbb{R}_+^*$:

$$
\begin{aligned}
\mathcal{N} &= \{(k, H_{\mathsf{ext}}) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \text{ s.t. } \mathcal{E}_{k,H_{\mathsf{ext}}} = E_{\mathcal{N}}\} \ , \\
\mathcal{P} &= \{(k, H_{\mathsf{ext}}) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \text{ s.t. } \mathcal{E}_{k,H_{\mathsf{ext}}} = E_{\mathcal{P}}\} \ , \\
\mathcal{M} &= \{(k, H_{\mathsf{ext}}) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \text{ s.t. } \mathcal{E}_{k,H_{\mathsf{ext}}} < \inf(E_{\mathcal{P}}, E_{\mathcal{N}})\} \ .
\end{aligned}
$$

The set $\mathcal{M}$ is the complementary of $\mathcal{P} \cup \mathcal{N}$ in $\mathbb{R}_+^* \times \mathbb{R}_+^*$; if $(k, H_{\mathsf{ext}}) \in \mathcal{M}$, then the superconductor is said to be in a *mixed state*.

Using this simple modelization we were able (see [Du99] and [Du01]) to prove following *monotonicity theorem*.

Theorem 1 *(i) If $(k, H_{\mathsf{ext}}) \in \mathcal{P}$, $k' \leq k$ and $H'_{ext} \leq H_{\mathsf{ext}}$ then $(k', H'_{ext}) \in \mathcal{P}$.*
(ii) If $(k, H_{\mathsf{ext}}) \in \mathcal{N}$, $k' \geq k$ and $H'_{ext} \geq H_{\mathsf{ext}}$ then $(k', \frac{k'}{k} H'_{ext}) \in \mathcal{N}$.

The existence of such a Theorem is possible only because the system is invariant by homotheties (see, for example, [DH] for the case of a superconductor restricted to a domain $\mathcal{D}$ of $\mathbb{R}^2$).

From this theorem we derived the existence of two functions $k \mapsto H_{c1}(k)$ and $k \mapsto H_{c2}(k)$ such that

$$
\begin{aligned}
\mathcal{N} &= \{(k, H_{\mathsf{ext}}), \text{ s.t. } H_{\mathsf{ext}} \geq H_{c2}(k)\}, \\
\mathcal{P} &= \{(k, H_{\mathsf{ext}}), \text{ s.t. } H_{\mathsf{ext}} \leq H_{c1}(k)\}, \\
\mathcal{M} &= \{(k, H_{\mathsf{ext}}), \text{ s.t. } H_{c1}(k) < H_{\mathsf{ext}} < H_{c2}(k)\} \ .
\end{aligned}
$$

Using this modelization we obtained in [Du01] the qualitative form of the phase diagram depicted in Figure 1, which is recalled in Section IV

This phase diagram is made of three curves:

(i) (*boundary normal-pure*) $H_{\mathsf{ext}} = H_{c1}(k) = H_{c2}(k) = \frac{1}{\sqrt{2}}$ with $k \leq \frac{1}{\sqrt{2}}$,

(ii) (*boundary normal-mixed*) $H_{\mathsf{ext}} = H_{c2}(k) = k$ with $k \geq \frac{1}{\sqrt{2}}$,

(iii) (*boundary pure-mixed*) $H_{\mathsf{ext}} = H_{c1}(k)$ with $k \geq \frac{1}{\sqrt{2}}$.

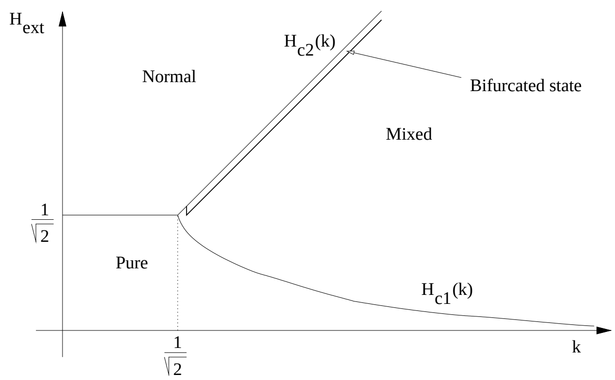

Figure 1: Phase diagram in Abrikosov modelization

The exact expression of curve (iii) is unknown. Those three curves meet at the triple point $k = H_{\mathsf{ext}} = \frac{1}{\sqrt{2}}$. A key point of the proof is that the case $k = \frac{1}{\sqrt{2}}$ is exactly solvable thanks to the Bochner-Kodaira-Nakano formula explained in Section III. Using a more advanced analysis of the case $k = \frac{1}{\sqrt{2}}$ in Section V, we prove in Section VI the following Theorem:

Theorem 2 *(i) There exist $\delta > 0$ and $S > 0$ such that for all h in $[0, \delta]$, we have*

$$-h \leq H_{c1}(\frac{1}{\sqrt{2}} + h) - \frac{1}{\sqrt{2}} \leq -Sh \ .$$

(ii) The critical magnetic field $H_{c1}(k)$ is strictly decreasing at $k = \frac{1}{\sqrt{2}}$.

II THE CHANGE OF VARIABLE

In this Section, we recall the original formulation of the problem by V.Ginzburg and L.Landau in [GL50] and how it is related to our formulation. They proposed the following expression for the density of energy in superconductors

$$\frac{1}{2}\|ik^{-1}\nabla\phi + \mathbf{A}\phi\|^2 + \frac{1}{4}(1 - |\phi|^2)^2 + \frac{1}{2}(\text{curl } \mathbf{A} - H_{\mathsf{ext}})^2$$

This expression belongs to $L^1_{\mathsf{loc}}(\mathbb{R}^3)$ if $(\phi, \mathbf{A})$ is in the Sobolev space $H^1_{\mathsf{loc}}(\mathbb{R}^3; \mathbb{C}) \times H^1_{\mathsf{loc}}(\mathbb{R}^3; \mathbb{R}^3)$. It is invariant under the gauge transform

$(\phi', \mathbf{A}') = (\phi e^{ikg}, \mathbf{A} + \nabla g)$ with $g \in H^2_{\mathsf{loc}}(\mathbb{R}^2)$; this property is shared by other physically significant quantities like the density of superconducting electron $|\phi|^2$, the magnetic field curl $\mathbf{A}$ and the current vector of superconducting electron $Re[\overline{\phi}(ik^{-1}\nabla\phi + \mathbf{A}\phi)]$.

We assumed that the problem is invariant under translation along the z-axis. This means that we consider pairs $(\phi, \mathbf{A})$, which satisfies: for every $h \in \mathbb{R}$, the pair $(\phi, \mathbf{A})(x_1, x_2, x_3 + h)$ is gauge equivalent to the pair $(\phi, \mathbf{A})$.

In fact, as proved in [Du99] p. 17, we can assume that the pairs $(\phi, \mathbf{A})$ considered are independent of x_3 and satisfy $\mathbf{A}_{x_3} = 0$. So, we can reduce ourself to a 2-dimensional problem.

We take $\mathcal{L}$ a 2-dimensional lattice of $\mathbb{R}^2$ with fundamental domain Ω of area 1. We consider the dilated lattice : $\mathcal{L}_\lambda = \sqrt{\lambda}\mathcal{L}$ with fundamental domain $\Omega_\lambda = \sqrt{\lambda}\Omega$. Following Abrikosov, we choose λ in $\mathbb{R}_+$ and restrict the analysis to pairs $(\phi, \mathbf{A})$, which are gauge periodic with respect to $\mathcal{L}_\lambda$ ([Ab]). This means that, for all $v \in \mathcal{L}_\lambda$, there exists $g^v \in H^2_{\mathsf{loc}}(\mathbb{R}^2)$ such that

$$\phi(z+v) = e^{ikg^v(z)}\phi(z) \quad \text{and} \quad \mathbf{A}(z+v) = \mathbf{A}(z) + \nabla g^v(z) \ .$$

Consequently, all the considered physical quantities are $\mathcal{L}_\lambda$-periodic. We denote by $|\Omega_\lambda|$ the area of Ω_λ, which is actually equal to λ.

A classic consequence (see [Du99], [BGT]) of gauge periodicity is that there exist $d \in \mathbb{Z}$ satisfying to

$$2\pi d = k \int_{\Omega_\lambda} \text{curl } \mathbf{A} \ .$$

We will then, according to Abrikosov, fix the quantization d per unit cell equal to 1.

The Ginzburg-Landau functional is obtained by integration of the local density over the fundamental domain Ω_λ and division by $|\Omega_\lambda|$. This gives:

$$F(\phi, \mathbf{A}) \ = \frac{1}{|\Omega_\lambda|}\int_{\Omega_\lambda} \frac{1}{2}\|ik^{-1}\nabla\phi + \mathbf{A}\phi\|^2 + \frac{1}{4}(1-|\phi|^2)^2 + \frac{1}{2}(\text{curl } \mathbf{A} - H_{\mathsf{ext}})^2,$$

which should be understood as a mean energy.

We denote by $H_{\mathsf{int}} = \frac{1}{|\Omega_\lambda|}\int_{\Omega_\lambda} \text{curl } \mathbf{A}$ the mean internal magnetic field induced by $\mathbf{A}$. The quantization relation is then rewritten as $2\pi = k\lambda H_{\mathsf{int}}$.

It is also a classical result (see [BGT], [YS] or [Du99], p. 21-29) that we can associate to the pair $(\phi, \mathbf{A})$, another pair $(\phi', \mathbf{A}')$, with the same Ginzburg-Landau energy but satisfying to

(i) $\mathbf{A}' = \frac{H_{\mathsf{int}}}{2\pi}\mathbf{A}_0 + \mathbf{P}$ with $\mathbf{P}$ $\mathcal{L}_\lambda$-periodic, div $\mathbf{P} = 0$, $\int_{\Omega_\lambda} \mathbf{P} = 0$,

(ii) $\phi'(z+v) = e^{ikg^v(z)}\phi'(z)$ with $g^v(x,y) = \frac{H_{\mathsf{int}}}{2}(v_x y - v_y x)$ for all $v \in \mathcal{L}_\lambda$.

This reduction is rather involved and is performed by a suitable gauge transform and a translation in x, y. The relation relating $\phi'(z+v)$ to $\phi'(z)$ actually defines the sections of a complex line bundle over the torus $\mathbb{R}^2/\mathcal{L}$; above result is so, a classification result.

With this expression one gets

$$\frac{1}{|\Omega_\lambda|}\int_{\Omega_\lambda} \frac{1}{2}(\text{curl }\mathbf{A} - H_{\mathsf{ext}})^2 = \frac{1}{|\Omega_\lambda|}\int_{\Omega_\lambda} \frac{1}{2}(\text{curl }\mathbf{P})^2 + \frac{1}{2}(H_{\mathsf{int}} - H_{\mathsf{ext}})^2.$$

This leads to the simple expression $F(\phi, \mathbf{A}) = F^{int}(\phi, \mathbf{P}) + \frac{1}{2}(H_{\mathsf{int}} - H_{\mathsf{ext}})^2$ with

$$F^{int}(\phi, \mathbf{P}) \;=\; \tfrac{1}{\lambda}\int_{\Omega_\lambda} \tfrac{1}{2}\|ik^{-1}\nabla\phi + \mathbf{A}\phi\|^2 + \tfrac{1}{4}(1 - |\phi|^2)^2 + \tfrac{1}{2}(\text{curl }\mathbf{P})^2.$$

The functional F^{int} is called internal energy and depends only on H_{int}, k, ϕ and $\mathbf{P}$.

The quantities H_{int}, k and λ are related by the quantization relation $2\pi = k\lambda H_{\mathsf{int}}$, which makes the analysis of F^{int} cumbersome. So, we reduce the complexity of the computation by the following change of variables and of functions:

$$\begin{cases} u(x) &= \phi(x\sqrt{\frac{2\pi}{kH_{\mathsf{int}}}})\ , \\ \mathbf{a}(x) &= \sqrt{\frac{2\pi k}{H_{\mathsf{int}}}}[\mathbf{A} - \frac{H_{\mathsf{int}}}{2\pi}\mathbf{A}_0](x\sqrt{\frac{2\pi}{kH_{\mathsf{int}}}}) = \sqrt{\frac{2\pi k}{H_{\mathsf{int}}}}\mathbf{A}(x\sqrt{\frac{2\pi}{kH_{\mathsf{int}}}}) - \mathbf{A}_0(x)\ . \end{cases}$$

We then obtain the formulation given in the introduction since the pair $(u, \mathbf{a})$ so defined belongs to $\mathcal{A}$ and verifies $E_{k,H_{\mathsf{int}}}(u, \mathbf{a}) = F^{int}(\phi, \mathbf{P})$.

III THE FUNCTIONAL $E_{k,H_{\mathsf{int}}}$

Let us now analyze the functional $E_{k,H_{\mathsf{int}}}$ by assuming here that k and H_{int} are fixed.

$E_{k,H_{\mathsf{int}}}$ is defined over $\mathcal{A}$ since $(u, \mathbf{a})$ of class H^1 guarantees local integrability of the density, while the compactness of the torus $\mathbb{R}^2/\mathcal{L}$ guarantees its integrability.

In fact, the variational theory of the functional $E_{k,H_{\mathsf{int}}}$ is easy (see [Du99]) since the torus $\mathbb{R}^2/\mathcal{L}$ is compact and the non-linear partial differential equations obtained for the critical points are elliptic; the vector bundle adds only technical difficulties (see [LM]). More precisely one can prove successively that:

1. *Coerciveness*: for every $C \in \mathbb{R}$ there is a $C' > 0$ such that $E_{k,H_{\mathsf{int}}}(u, \mathbf{a}) < C$ implies $\|u\|_{H^1} + \|\mathbf{a}\|_{H^1} \leq C'$.

2. *Lower semicontinuity*: If $(u_n, \mathbf{a}_n) \in \mathcal{A}$ converges weakly to $(u, \mathbf{a}) \in \mathcal{A}$, then $E_{k,H_{\mathsf{int}}}(u, \mathbf{a}) \leq \underline{\lim}_n E_{k,H_{\mathsf{int}}}(u_n, \mathbf{a}_n)$.

3. *Minimum*: The functional $E_{k,H_{\mathsf{int}}}$ attains its minimum on at least one pair $(u, \mathbf{a}) \in \mathcal{A}$.

4. *Ginzburg-Landau equations*: The minimizing pairs satisfy to the following equation

$$\left\{ \begin{array}{rcl} \mu[i\nabla + \mathbf{A}_0 + \mathbf{a}]^2 u & = & (1 - |u|^2)u \\ \Delta \mathbf{a} & = & \frac{1}{k^2} \operatorname{Re}[\overline{u}(i\nabla u + (\mathbf{A}_0 + \mathbf{a})u)] \end{array} \right.$$

5. *Regularity*: The pairs $(u, \mathbf{a}) \in \mathcal{A}$ verifying the Ginzburg-Landau equations are in fact of class C^∞.

6. *Maximum principle*: The pairs $(u, \mathbf{a}) \in \mathcal{A}$ verifying the Ginzburg-Landau equations satisfy $|u| \leq 1$.

We now explain the Bochner-Kodaira-Nakano formula for the functional $E_{k,H_{\mathsf{int}}}$ (see [De], [JaTa] and [WY] for related formulas and results). This classical formula is also called Bogmol'nyi formula, Weitzenbock formula, Lichnerowicz formula (see [JT]) according to different scientific schools.

We set $\mathbf{C} = \mathbf{A}_0 + \mathbf{a}$; we get curl $\mathbf{C} = 2\pi +$ curl $\mathbf{a}$ and define

$$A_{+,H_{\mathsf{int}}}(u, \mathbf{a}) = \int_\Omega \frac{\mu}{2} |D_+ u|^2 + \frac{1}{4} |\mu \operatorname{curl} \mathbf{C} - (1 - |u|^2)|^2,$$

where $\mu = \frac{H_{\mathsf{int}}}{2\pi k}$ and $D_+ = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} + C_y - iC_x$.

Theorem 3 *(*Bochner-Kodaira-Nakano*)*
For all $(u, \mathbf{a}) \in \mathcal{A}$, we have :

$$E_{\frac{1}{\sqrt{2}},H_{\mathsf{int}}}(u, \mathbf{a}) = \mu\pi - (\mu\pi)^2 + A_{+,H_{\mathsf{int}}}(u, \mathbf{a}).$$

Proof. We perform computations with smooth functions and then extend by density. After expansion, simplification and regrouping one obtains

$$\begin{array}{rcl}\{A_{+,H_{\mathsf{int}}}-E_{\frac{1}{\sqrt{2}},H_{\mathsf{int}}}\}(u,\mathbf{a}) & = & \frac{1}{2}\int_\Omega \operatorname{div}\,\mathbf{W}-\mu\operatorname{curl}\,\mathbf{C}\\ & + & \frac{\mu^2}{4}\int_\Omega |\operatorname{curl}\,\mathbf{C}|^2-|\operatorname{curl}\,\mathbf{a}|^2\end{array}$$

with $\mathbf{W}=\begin{pmatrix}\overline{u}(i\frac{\partial u}{\partial y}+C_y u)\\ -\overline{u}(i\frac{\partial u}{\partial x}+C_x u)\end{pmatrix}$. The vector field $\mathbf{W}$ being $\mathcal{L}$-periodic, the integral of its divergence over Ω is 0. The formula is then obtained by replacing curl $\mathbf{C}$ by $2\pi+\operatorname{curl}\,\mathbf{a}$ and using $\int_\Omega \operatorname{curl}\,\mathbf{a}=0$. $\square$

The *magnetic Schrodinger* operator is defined as $H=[i\nabla+\mathbf{A}_0]^2$; its spectrum, called *Landau levels*, is recalled in next theorem.

Theorem 4 *(i) The operator H admits a self-adjoint extension over $L^2(E_1)$, also denoted by H, whose domain is $H^2(E_1)$.*

(ii) It can be expressed as $H=L_+^*L_++2\pi$ with $[L_+,L_+^*]=4\pi$ and $L_+=2\partial_{\overline{z}}+\pi z$.

(iii) Its spectrum is discrete, $sp(H)=2\pi+4\pi\mathbb{N}$, and every eigenvalue is simple.

(iv) The eigenvector u_0 associated to $\lambda=2\pi$ satisfies $L_+(u_0)=0$ and has a unique simple zero in Ω denoted by z_0.

Proof. (i) and the discreteness of the spectrum follow from the fact that H is an elliptic pseudo-differential operator of order 2 defined over the vector bundle of a compact manifold (see [LM]).

Formula $H=L_+^*L_++2\pi$ and $[L_+,L_+^*]=4\pi$ are proved by first computing with smooth functions and then extending by density.

If we proved that the equation $L_+(u)=0$ has a unique solution u_0 up to scalar, then by the harmonic oscillator formalism we would get (iii).

In fact, if one writes, $u_0(z)=e^{-|z|^2\frac{\pi}{2}}s(z)$, then $s(z)$ is analytic. Furthermore, without loss of generality, we can assume that $\mathcal{L}$ is generated by the vectors $v_1=(u,0)$ and $v_2=(w,r)$ with $ru=1$. Then, after using gauge periodicity conditions, one finds the following expression for u_0:

$$u_0(x,y)=e^{i\pi xy}\sum_{n\in\mathbb{Z}}e^{-\pi(y+nu)^2}e^{\pi n^2 iwu+2\pi nuix}\ .$$

This expression is a theta function; it is known that such function have a unique simple zero in Ω (see [Cha]). Another method of proof is the use of Rouché Theorem as done in [Du99]. $\square$

Theorem 5 *If $k \geq \frac{1}{\sqrt{2}}$ and $H_{\mathsf{int}} \geq k$, then $m_E(k, H_{\mathsf{int}}) = \frac{1}{4}$. Furthermore, the minimum is met only by the pair $(0,0)$.*

Proof. We use following expansion of the functional $E_{k,H_{\mathsf{int}}}$:

$$\begin{aligned}
E_{k,H_{\mathsf{int}}}(u,\mathbf{a}) &\geq E_{\frac{1}{\sqrt{2}},H_{\mathsf{int}}}(u,\mathbf{a}) \\
&\geq (\mu\pi) - (\mu\pi)^2 + \textstyle\int_\Omega \frac{\mu}{2}|D_+u|^2 + \frac{1}{4}|2\mu\pi - 1 + \mu\,\mathrm{curl}\,\mathbf{a} + |u|^2|^2 \\
&\geq (\mu\pi) - (\mu\pi)^2 + \frac{(2\mu\pi-1)^2}{4} + \frac{1}{4}\textstyle\int_\Omega 2(2\mu\pi-1)(\mu\,\mathrm{curl}\,\mathbf{a} + |u|^2) \\
&\quad + \frac{1}{4}\textstyle\int_\Omega |\mu\,\mathrm{curl}\,\mathbf{a} + |u|^2|^2 \\
&\geq \frac{1}{4} + \frac{2\mu\pi - 1}{2}\textstyle\int_\Omega |u|^2\ .
\end{aligned}$$

Then using the hypothesis $2\mu\pi - 1 = \frac{H_{\mathsf{int}}}{k} - 1 \geq 0$, we get $m_E(k, H_{\mathsf{int}}) \geq \frac{1}{4}$ by positivity of terms of above equation.

Now assume that $E_{k,H_{\mathsf{int}}}(u,\mathbf{a}) = \frac{1}{4}$; in fact, last computation give us the following equalities:

$$\left\{\begin{array}{lllllll}
0 &=& (2\mu\pi - 1)\int_\Omega |u|^2, & & 0 &=& \int_\Omega |\,\mathrm{curl}\,\mathbf{a} + |u|^2|^2, \\
0 &=& (k^2 - \frac{1}{2})\int_\Omega |\,\mathrm{curl}\,\mathbf{a}|^2, & & 0 &=& \int_\Omega |D_+u|^2.
\end{array}\right.$$

The second equality give us $\mathrm{curl}\,\mathbf{a} + |u|^2 = 0$, which integrated over Ω yields

$$\int_\Omega |u|^2 = -\int_\Omega \mathrm{curl}\,\mathbf{a} = 0$$

and then $u = 0$.

Now, using the equation $\mathrm{div}\,\mathbf{a} = 0$, one obtains the equality $\mathrm{curl}^*\,\mathrm{curl}\,\mathbf{a} = \Delta\,\mathbf{a} = 0$. The potential vector $\mathbf{a}$ is $\mathcal{L}$ periodic; so, it has to be constant. Now, the property $\int_\Omega \mathbf{a} = 0$ yields $\mathbf{a} = 0$. $\square$

IV THE PHASE DIAGRAM

Let us first consider the special case when $k = H_{\mathsf{ext}} = \frac{1}{\sqrt{2}}$. We have the following Lemma:

Lemma 6 *One has*

(i) $E_{\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}}(H_{\mathsf{int}}, u, \mathbf{a}) = \frac{1}{4} + A_{+,H_{\mathsf{int}}}(u,\mathbf{a})$,

(ii) $\mathcal{E}_{\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}} = \frac{1}{4}$,

(iii) $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \in \mathcal{P} \cap \mathcal{N}$.

Proof. (i) is in fact a rewriting of the Bochner-Kodaira-Nakano formula; it yields (ii) by positivity of $A_{+,H_{\mathsf{int}}}$, while (iii) is obtained by remarking that $E_{\mathcal{N}} = \frac{1}{4} = \frac{1}{2}(\frac{1}{\sqrt{2}})^2 = E_{\mathcal{P}}$. □

Theorem 7 *(Type I superconductors) If $k \le \frac{1}{\sqrt{2}}$, then:*

(i) If $H_{\mathsf{ext}} \le \frac{1}{\sqrt{2}}$, then $\mathcal{E}_{k,H_{\mathsf{ext}}} = E_{\mathcal{P}}$ and $(k, H_{\mathsf{ext}}) \in \mathcal{P}$,

(ii) If $H_{\mathsf{ext}} \ge \frac{1}{\sqrt{2}}$, then $\mathcal{E}_{k,H_{\mathsf{ext}}} = E_{\mathcal{N}}$ and $(k, H_{\mathsf{ext}}) \in \mathcal{N}$.

Proof. Lemma 6 combined with Theorem 1.(i) give the result in the case $H_{\mathsf{ext}} \le \frac{1}{\sqrt{2}}$.

In particular, if $k \le \frac{1}{\sqrt{2}}$ we have $(k, \frac{1}{\sqrt{2}}) \in \mathcal{P}$ and so, $\mathcal{E}_{k,\frac{1}{\sqrt{2}}} = E_{\mathcal{P}} = \frac{1}{2}(\frac{1}{\sqrt{2}})^2 = \frac{1}{4} = E_{\mathcal{N}}$; therefore Theorem 1.(ii) gives the conclusion in case $H_{\mathsf{ext}} \ge \frac{1}{\sqrt{2}}$. □

Theorem 8 *(Type II superconductors) If $H_{\mathsf{ext}} \ge k \ge \frac{1}{\sqrt{2}}$, then:*

(i) If $H_{\mathsf{ext}} \ge k$, then $\mathcal{E}_{k,H_{\mathsf{ext}}} = E_{\mathcal{N}}$ and $(k, H_{\mathsf{ext}}) \in \mathcal{N}$,

(ii) If $H_{\mathsf{ext}} < k$, then $(k, H_{\mathsf{ext}}) \notin \mathcal{N}$.

Proof. Lemma 6 combined with Theorem 1.(ii) gives (i).

By setting $H_{\mathsf{int}} = H_{\mathsf{ext}}$, $u = \alpha u_0$, $\mathbf{a} = 0$ and doing a development of order 2 around the pair $(0, 0)$, one obtains

$$E_{k,H_{\mathsf{ext}}}(H_{\mathsf{ext}}, \alpha u_0, 0) = E_{k,H_{\mathsf{ext}}}(\alpha u_0, 0) = \frac{1}{4} + \frac{1}{2}(\frac{H_{\mathsf{ext}}}{k} - 1)\alpha^2 + o(\alpha^2) \ .$$

Since $k > H_{\mathsf{ext}} = H_{\mathsf{int}}$, one obtains for α small $E_{k,H_{\mathsf{ext}}}(H_{\mathsf{ext}}, \alpha u_0, 0) < \frac{1}{4}$; so, the energy will be lower than $\frac{1}{4}$, i.e. $(k, H_{\mathsf{ext}}) \notin \mathcal{N}$. □

V ANALYSIS OF THE CASE $k = \frac{1}{\sqrt{2}}$

In this section we will find all pairs $(u, \mathbf{a})$ verifying $A_{+,H_{\mathsf{int}}}(u, \mathbf{a})$, thus get the value of $m_E(\frac{1}{\sqrt{2}}, H_{\mathsf{int}})$. A similar study is done in [Al2] for a rectangular problem. In book [JaTa], the case considered is of u defined over $\mathbb{R}^2$, while in paper ([Ga]) the problem is considered over a Riemann surface. Also, in

[JaTa] it is proved that all critical points of the Ginzburg-Landau functional are solution of the Bogmol'nyi equations, but their proof does not apply to our case.

The papers ([KW]), ([CY]), ([WY]) are devoted to existence theorem concerning the Kazdan-Warner equation. They get as a byproduct existence Theorems for the self-dual equations.

Theorem 9 *(Kazdan-Warner, see [KW]) If h is a positive function, $h \neq 0$, and $C^\infty(\mathbb{R}^2/\mathcal{L})$. If $A > 0$ then the equation*

$$-\Delta f + e^f h = A$$

has a unique solution f in $C^\infty(\mathbb{R}^2/\mathcal{L})$.

We define

$$\left\{ \begin{array}{rcl} u_{H_{\mathsf{int}}} & = & u_0 e^{f_{H_{\mathsf{int}}}} \\ \mathbf{a}_{\mathbf{H}_{\mathsf{int}}} & = & \left(\frac{\partial f_{H_{\mathsf{int}}}}{\partial y}, -\frac{\partial f_{H_{\mathsf{int}}}}{\partial x}\right), \end{array} \right.$$

with $f_{H_{\mathsf{int}}}$ being the unique solution of $1-2\mu\pi = |u_0|^2 e^{2f} - \mu\Delta f$ and $\mu = \frac{H_{\mathsf{int}}}{\pi\sqrt{2}}$.

Let us introduce first the following family of sections of E_1:

$$u_h(x, y) = e^{i\pi(h_y x - h_x y)} u_0(z - h) \ .$$

Recall that z_0 is the zero of u_0 in $\mathbb{R}^2/\mathcal{L}$; the section u_h verifies the following easy properties

$$\left\{ \begin{array}{l} u_h \in C^\infty(E_1), \qquad L_+(u_h) = 2\pi h u_h, \\ u_h(z) = 0 \text{ if and only if } z \in z_0 + h + \mathcal{L} \ . \end{array} \right.$$

Furthermore, for any $h \in \mathbb{R}^2$, $v \in \mathcal{L}$, there exists $\alpha \in \mathbb{R}$ such that

$$u_{h+v}(z) = e^{i\alpha} e^{2i\pi(v_y x - v_x y)} u_h(z) \ .$$

Theorem 10 *We assume $H_{\mathsf{int}} \leq \frac{1}{\sqrt{2}}$.*

(i) If $(u, \mathbf{a}) \in \mathcal{A}$ satisfies $A_{+,H_{\mathsf{int}}}(u, \mathbf{a}) = 0$, then there exist $c \in \mathbb{R}$ such that $(u, \mathbf{a}) = (e^{ic} u_{H_{\mathsf{int}}}, \mathbf{a}_{\mathbf{H}_{\mathsf{int}}})$.

(ii) The pair $(u_{H_{\mathsf{int}}}, \mathbf{a}_{\mathbf{H}_{\mathsf{int}}})$ satisfies to

$$\left\{ \begin{array}{l} \int_\Omega (1 - |u_{H_{\mathsf{int}}}|^2)^2 = \mu^2[(2\pi)^2 + \int_\Omega |\operatorname{curl} \mathbf{a}_{\mathbf{H}_{\mathsf{int}}}|^2] \\ \int_\Omega \frac{\mu}{2} \|i\nabla u_{H_{\mathsf{int}}} + (\mathbf{A}_0 + \mathbf{a}) u_{H_{\mathsf{int}}}\|^2 + \frac{\mu^2}{2} |\operatorname{curl} \mathbf{a}_{H_{\mathsf{int}}}|^2 = (\mu\pi) - 2(\mu\pi)^2 \ . \end{array} \right.$$

Proof. Let $(u, \mathbf{a}) \in \mathcal{A}$ be a pair satisfying $A_{+,H_{\text{int}}}(u, \mathbf{a}) = 0$, it then verifies the following Bogmol'nyi equations

$$D_+ u = L_+ u + (a_y - i a_x) u = 0 \text{ and } 2\mu\pi + \mu \operatorname{curl} \mathbf{a} = 1 - |u|^2$$

and, by Theorem 3, minimizes the functional $E_{\frac{1}{\sqrt{2}}, H_{\text{int}}}$. Therefore, by Section III it satisfies the Ginzburg-Landau equations and so, it is C^∞.

Since the vector bundle E_1 is non trivial the section u possess at least one zero in $\mathbb{R}^2/\mathcal{L}$, which we write as $z_h = z_0 + h$.

The zero-set of the function u defined on $\mathbb{R}^2$ contains $z_h + \mathcal{L}$, while the zero-set of u_h is exactly $z_h + \mathcal{L}$; so, one defines on $\mathbb{R}^2 - (z_h + \mathcal{L})$ the function

$$f = \frac{u}{u_h} \, .$$

Since both u and u_h are section of the vector bundle E_1, the function f is $\mathcal{L}$-periodic. The equation $D_+ u = 0$ is rewritten on $\mathbb{R}^2 - (z_h + \mathcal{L})$ as:

$$0 = 2(\partial_{\overline{z}} f) u_h + f D_+ u_h = 2(\partial_{\overline{z}} f) u_h + [2\pi h f + (a_y - i a_x) f] u_h \, ,$$

Since u_h is not zero on $\mathbb{R}^2 - (z_h + \mathcal{L})$ we obtain:

$$\partial_{\overline{z}} f = f w \text{ with } w = \frac{1}{2}[(-a_y - 2\pi h_x) + i(a_x - 2\pi h_y)] \, .$$

Note that the function w is defined on $\mathbb{R}^2$, also it is C^∞ and $\mathcal{L}$-periodic.

We now want to extend f to $\mathbb{R}^2$: it is a classic result of complex analysis that the equation $\partial_{\overline{z}} k = w$ has a C^∞ solution k on $\mathbb{R}^2$.

The function $g = f e^{-k}$ is defined on $\mathbb{R}^2 - (z_h + \mathcal{L})$, satisfies $\partial_{\overline{z}} g = 0$ and is so, analytic. If $m \in z_h + \mathcal{L}$ then $u = O(z - m)$, since u is C^∞. The complex m is a simple zero of u_h, consequently $u_h^{-1} = O(|z - m|^{-1})$ and $f = O(1)$ at m.

The function g stay bounded around m and is analytic outside m. By a classic result of complex analysis, we get that g can be extended to m in a complex analytic function. The function g is extended to $\mathbb{C}$ and so, f too.

The function g is analytic and so, its zero set is discrete. There exist a translate Ω' of Ω such that the boundary $\partial\Omega'$ of Ω' does not meet any zero of g.

By Rouch theorem the number n of zero of g in Ω' is equal to :

$$n = \frac{1}{2\pi i} \int_{\partial\Omega'} \frac{\partial_z g}{g} dz = \frac{1}{2\pi i} \int_{\partial\Omega'} \frac{\partial_z f}{f} dz - \frac{1}{2\pi i} \int_{\partial\Omega'} \partial_z k dz = \frac{-1}{2\pi i} \int_{\partial\Omega'} \partial_z k dz.$$

The integral of $\frac{\partial_z f}{f}$ over $\partial\Omega'$ is zero, since f is $\mathcal{L}$-periodic.

Now using Stokes theorem, we get :

$$\begin{aligned} n &= \tfrac{-1}{2\pi i}\int_{\partial\Omega'}\partial_z k dz = \tfrac{-1}{2\pi i}\int_{\Omega'} d(\partial_z k dz) = \tfrac{-1}{2\pi i}\int_{\Omega'}\partial_{\overline{z}}\partial_z k d\overline{z}\wedge dz \\ &= \tfrac{-1}{2\pi i}\int_{\Omega'}\partial_z\partial_{\overline{z}} k d\overline{z}\wedge dz = \tfrac{-1}{2\pi i}\int_{\Omega'}\partial_z w d\overline{z}\wedge dz. \end{aligned}$$

The function w is $\mathcal{L}$-periodic; consequently the function $\partial_z w$ is a $\mathcal{L}$-periodic function, which has integral zero over Ω'; so, $n = 0$.

Since $f = ge^k$, the function f has no zero over $\mathbb{R}^2$. Since $\mathbb{R}^2$ is simply connected there exist a complex valued C^∞ function ψ such that $f = e^\psi$.

The function ψ is not $\mathcal{L}$-periodic, but since the function f is $\mathcal{L}$-periodic and C^∞ there exist two integer n_1, n_2 such that

$$\psi(z + v_1) = \psi(z) + 2\pi i n_1 \text{ and } \psi(z + v_2) = \psi(z) + 2\pi i n_2 .$$

We pose $v' = n_1 v_2 - n_2 v_1$, the function $\psi_2(z) = \psi(z) - 2\pi i \det(z, v')$ is $\mathcal{L}$-periodic, and we have :

$$u(z) = f(z)u_h(z) = e^{\psi_2(z)+2\pi i[xv'_y - yv'_x]}u_h(z) = e^{\psi_2(z)-i\alpha}u_{h+v'}(z)$$

with $\alpha \in \mathbb{R}$. We set $h_3 = h + v'$, $\psi_3 = \psi_2 - i\alpha$, and we rewrite u as:

$$u(z) = e^{\psi_3(z)}u_{h_3}(z)$$

with ψ_3 a $\mathcal{L}$-periodic C^∞ function. The Bogmol'nyi equations are rewritten as

$$\begin{cases} \frac{\partial\psi_3}{\partial\overline{z}} &= \frac{1}{2}[(-a_y - \pi h_{3,x}) + i(a_x - \pi h_{3,y})] , \\ 0 &= 2\mu\pi - 1 + |u_{h_3}|^2 e^{2\,\mathrm{Re}\,\psi_3} + \mu\,\mathrm{curl}\,\mathbf{a}. \end{cases}$$

The real and imaginary part of first equation give us the expression of the potential vector:

$$\begin{cases} a_x &= \pi h_{3,y} + \frac{\partial\,\mathrm{Re}\,\psi_3}{\partial y} + \frac{\partial\,\mathrm{Im}\,\psi_3}{\partial x} , \\ a_y &= -\pi h_{3,x} - \frac{\partial\,\mathrm{Re}\,\psi_3}{\partial x} + \frac{\partial\,\mathrm{Im}\,\psi_3}{\partial y} . \end{cases}$$

The equation div $\mathbf{a} = 0$ is then rewritten as $\Delta\,\mathrm{Im}\,\psi_3 = 0$. Thus $\mathrm{Im}\,\psi_3$ is constant, since it is $\mathcal{L}$-periodic. We now write $\psi_3 = f + ic$ with f a real C^∞, $\mathcal{L}$-periodic function; so, one has

$$a_x = \pi h_{3,y} + \frac{\partial f}{\partial y} \text{ and } a_y = -\pi h_{3,x} - \frac{\partial f}{\partial x} .$$

The functions $\mathbf{a}$, $\frac{\partial f}{\partial x}$, and $\frac{\partial f}{\partial y}$ have zero integral over Ω. So, we have $h_3 = 0$ and the zero of u in Ω is z_0.

One then obtain curl $\mathbf{a} = -\Delta f$ and the following equation for f:

$$0 = 2\mu\pi - 1 + |u_0|^2 e^{2f} - \mu\Delta f \ .$$

So, one gets $f = f_{H_{\mathsf{int}}}$; now above equation rewrites as

$$-\mu \operatorname{curl} \mathbf{a}_{\mathbf{H}_{\mathsf{int}}} = 2\mu\pi - 1 + |u_{H_{\mathsf{int}}}|^2 \ .$$

It yields $\int_\Omega |u_{H_{\mathsf{int}}}|^2 = 1 - 2\mu\pi$ and $\int_\Omega (1 - |u_{H_{\mathsf{int}}}|^2)^2 = \mu^2[(2\pi)^2 + \int_\Omega |\operatorname{curl} \mathbf{a}_{\mathbf{H}_{\mathsf{int}}}|^2]$, the second equation of (ii) is then obtained by Theorem 3. $\square$

Corollary 11 . *For every positive H_{int} one has:*

$$m_E(\frac{1}{\sqrt{2}}, H_{\mathsf{int}}) = \begin{cases} \frac{H_{\mathsf{int}}}{\sqrt{2}} - (\frac{H_{\mathsf{int}}}{\sqrt{2}})^2 & \text{if} \quad H_{\mathsf{int}} \leq \frac{1}{\sqrt{2}} \ , \\ \frac{1}{4} & \text{if} \quad H_{\mathsf{int}} \geq \frac{1}{\sqrt{2}} \ . \end{cases}$$

Proof. By Theorem 3, one has the inequality $m_E(\frac{1}{\sqrt{2}}, H_{\mathsf{int}}) \geq \frac{H_{\mathsf{int}}}{\sqrt{2}} - (\frac{H_{\mathsf{int}}}{\sqrt{2}})^2$, since $A_{+,H_{\mathsf{int}}} \geq 0$. This lower bound is attained by the pair $(u_{H_{\mathsf{int}}}, \mathbf{a}_{H_{\mathsf{int}}})$.

Theorem 5 give the result if $H_{\mathsf{int}} \geq \frac{1}{\sqrt{2}}$. $\square$

Remark 12 *It can be shown that the pair $(u_{H_{\mathsf{int}}}, \mathbf{a}_{\mathbf{H}_{\mathsf{int}}})$ depends continuously on H_{int} and vanish for $H_{\mathsf{int}} = \frac{1}{\sqrt{2}}$, i.e. it is a* bifurcated state *(see [Du99]).*

VI LOCAL STUDY

We define

$$\begin{aligned} H_k(u, \mathbf{a}) &= \tfrac{1}{4\pi k} \int_\Omega \|i\nabla u + (\mathbf{A}_0 + \mathbf{a})u\|^2 \\ &+ \sqrt{[\tfrac{1}{2} + \tfrac{1}{2(2\pi)^2} \int_\Omega |\operatorname{curl} \mathbf{a}|^2][\int_\Omega (1 - |u|^2)^2]} \ . \end{aligned}$$

Theorem 13 *If $k \geq \frac{1}{\sqrt{2}}$ then $H_{c1}(k) = \inf_{(u,\mathbf{a}) \in \mathcal{A}} H_k(u, \mathbf{a})$. If this infimum is attained on a pair, say, $(u', \mathbf{a}') \in \mathcal{A}$, then one has*

$$E_{k,H_{c1}(k)}(H_{\mathsf{int}}, u', \mathbf{a}') = \frac{H_{c1}^2(k)}{2} \quad \textit{with} \quad H_{\mathsf{int}} = \frac{1}{2}\sqrt{\frac{\int_\Omega (1 - |u'|^2)^2}{\frac{1}{2} + \frac{1}{2(2\pi)^2} \int_\Omega |\operatorname{curl} \mathbf{a}'|^2}} \ .$$

Proof. By Section I, we have $(k, H_{\mathsf{ext}}) \in \mathcal{P}$ equivalent to:

$$E_{k,H_{\mathsf{int}}}(u, \mathbf{a}) + \frac{1}{2}(H_{int} - H_{ext})^2 \geq \frac{H_{ext}^2}{2} \ ,$$

which after simplification is equivalent to

$$\begin{cases} H_{\mathsf{int}}[\frac{1}{2} + \frac{1}{2(2\pi)^2}\int_\Omega |\operatorname{curl} \mathbf{a}|^2] + \frac{1}{4H_{\mathsf{int}}}\int_\Omega (1-|u|^2)^2 \\ +\frac{1}{4\pi k}\int_\Omega \|i\nabla u + (\mathbf{A}_0 + \mathbf{a})u\|^2 \geq H_{ext}. \end{cases}$$

The minimum over $H_{\mathsf{int}} > 0$ of the above expression is attained for $H_{\mathsf{int}} = \frac{1}{2}\sqrt{\frac{\int_\Omega (1-|u|^2)^2}{\frac{1}{2}+\frac{1}{2(2\pi)^2}\int_\Omega |\operatorname{curl} \mathbf{a}|^2}}$ which yields the Theorem. $\square$

The above expression of $H_{c1}(k)$ allow us to obtain $H_{c1}(k) = O(\frac{\ln k}{k})$ (see [Du99]). From Theorem 7, one has $H_{c1}(\frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2}}$.

Theorem 14 *The set of pairs $(u, \mathbf{a}) \in \mathcal{A}$ verifying $H_{\frac{1}{\sqrt{2}}}(u, \mathbf{a}) = \frac{1}{\sqrt{2}}$ is*

$$(e^{ic} u_{H_{\mathsf{int}}}, \mathbf{a}_{\mathbf{H}_{\mathsf{int}}})$$

with $c \in \mathbb{R}$ and $0 < H_{\mathsf{int}} \leq \frac{1}{\sqrt{2}}$.

Proof. If $(u, \mathbf{a}) \in \mathcal{A}$ satisfies $H_{\frac{1}{\sqrt{2}}}(u, \mathbf{a}) = \frac{1}{\sqrt{2}}$, then one has

$$E_{\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}}(H_{\mathsf{int}}, u, \mathbf{a}) = \frac{1}{4} \quad \text{and} \quad H_{\mathsf{int}} = \frac{1}{2}\sqrt{\frac{\int_\Omega (1-|u|^2)^2}{\frac{1}{2} + \frac{1}{2(2\pi)^2}\int_\Omega |\operatorname{curl} \mathbf{a}|^2}} \ .$$

By Lemma 6.(i), first equation simplifies to $A_{+,H_{\mathsf{int}}}(u, \mathbf{a}) = 0$, and then using Theorem 10 to $(u, \mathbf{a}) = (e^{ic} u_{H_{\mathsf{int}}}, \mathbf{a}_{\mathbf{H}_{\mathsf{int}}})$.

When the expression of $(u, \mathbf{a})$ is substituted into the second equation, one obtains

$$4H_{\mathsf{int}}^2 = \frac{\int_\Omega (1-|u_{H_{\mathsf{int}}}|^2)^2}{\frac{1}{2} + \frac{1}{2(2\pi)^2}\int_\Omega |\operatorname{curl} \mathbf{a}_{\mathbf{H}_{\mathsf{int}}}|^2} \ .$$

By Theorem 10.(ii), this relation is always satisfied. $\square$

Theorem 15 *(i) There exist $\delta > 0$ and $S > 0$ such that for all h in $[0, \delta]$, we have*

$$-h \leq H_{c1}(\frac{1}{\sqrt{2}} + h) - \frac{1}{\sqrt{2}} \leq -Sh \ .$$

(ii) The critical magnetic field $H_{c1}(k)$ is strictly decreasing at $k = \frac{1}{\sqrt{2}}$.

Proof. The expression of $H_{c1}(k)$ obtained in Theorem 13 give us that the function $k \mapsto kH_{c_1}(k)$ is increasing; this yields the lower bound.

Now we will prove the upper bound by using the $(u_{H_{\mathsf{int}}}, \mathbf{a}_{\mathbf{H}_{\mathsf{int}}})$ as quasi-modes. If $k = \frac{1}{\sqrt{2}} + h$ then we will have

$$H_k(u_{H_{\mathsf{int}}}, \mathbf{a}_{\mathbf{H}_{\mathsf{int}}}) = \frac{1}{\sqrt{2}} - \frac{h}{2\pi} \int_\Omega \|i\nabla u_{H_{\mathsf{int}}} + (\mathbf{A}_0 + \mathbf{a}_{\mathbf{H}_{\mathsf{int}}}) u_{H_{\mathsf{int}}}\|^2 + o(h) \ .$$

We get the following values of S using Bochner-Kodaira-Nakano

$$\begin{array}{rcl} S & = & \sup_{0<H_{\mathsf{int}}<\frac{1}{\sqrt{2}}} \frac{1}{2\pi} \int_\Omega \|i\nabla u_{H_{\mathsf{int}}} + (\mathbf{A}_0 + \mathbf{a}_{\mathbf{H}_{\mathsf{int}}}) u_{H_{\mathsf{int}}}\|^2 \\ & = & \sup_{0<H_{\mathsf{int}}<\frac{1}{\sqrt{2}}} [1 - \frac{H_{\mathsf{int}}}{\frac{1}{\sqrt{2}}} - \frac{H_{\mathsf{int}}}{2\pi^2\sqrt{2}} \int_\Omega |\operatorname{curl} \mathbf{a}_{\mathbf{H}_{\mathsf{int}}}|^2] \end{array}$$

follows from Theorem 10.(ii). □

One may want now to know the exact value of S at $\frac{1}{\sqrt{2}}$. Using numerical simulations we obtain that the function

$$\chi(H_{\mathsf{int}}) = 1 - \sqrt{2} H_{\mathsf{int}} - \frac{H_{\mathsf{int}}}{2\pi^2\sqrt{2}} \int_\Omega |\operatorname{curl} \mathbf{a}_{\mathbf{H}_{\mathsf{int}}}|^2$$

is decreasing and has a limit of approximately 0.78 at $H_{\mathsf{int}} = 0$ for a square lattice.

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